Lecture 10: Infect (1/4)

How do epidemic and gossip reach people? (i.e., how computer viruses spread?)

COMS 4995-1: Introduction to Social Networks
Part II: Cascade
Tuesday, October 18th
What do these have in common?
Epidemic

* Sense and etymology: επι ‘upon’ + δεμος ‘people’
  – (1) spread from people to people accidentally
  – (2) significant size (i.e. constant fraction of nodes)

* Stronger variants:
  – Endemic=resides; Pandemic=everyone within reach
  – It shaped history, shapes our present:
    Black plague (XIII\textsuperscript{th}),
    America (XVI\textsuperscript{th}),
    AIDS (XX\textsuperscript{th})
Outline of Infect

- Continuous epidemics, “logistic model”
- Discrete epidemics, “graph”
- Epidemic algorithms
Dynamics of population growth

- reproduction
- access to resources (food)

\[ \frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \]
- \( r \): rate of reproduction,
- \( K \): # indiv. sustainable with available resources
Lemma: If dynamic system $y$ satisfies $\frac{dy}{dt} = \frac{c}{f(y)}$

- Then for any $t$, we have $F(y(t))-F(y(0)) = c \cdot t$
- where $F$ is a primitive function of $f$

\[
y: \quad \frac{dy}{dt} = \frac{c}{f(y)} \quad \int F \text{ any primitive of } f \quad F(t) = \int_0^t f(s) ds$

$F(y(t))-F(y(0))$
Assume \( r = K = 1 \) we have \[
\frac{d}{dt} P(t) = P(t)(1 - P(t)) = \frac{c}{f(P(t))}
\]

- \( P \) satisfies lemma with \( f(y) = \frac{1}{y} + \frac{1}{1-y} \)
- Hence \( P(t) = \frac{P(0)}{P(0) + (1-P(0)) e^{-t}} \)

\[
f(P) = \frac{1}{P(1-P)} ; \quad \frac{1}{P(1-P)} = \frac{1-P+p}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P} : \frac{1}{P} = \frac{1}{1-P}
\]

\[
F(p) = \ln(P) - \ln(1-P) = \ln \left( \frac{P}{1-P} \right)
\]

\[
F(P(t)) - F(P(0)) = 7x + t ; \quad \ln \left( \frac{P(t)}{1-P(t)} \right) - \ln \left( \frac{P(0)}{1-P(0)} \right) = \frac{c}{f(P(t))}
\]
Solution of logistic dynamics

\[ \ln \left( \frac{P(t)}{1-P(t)} \right) - \ln \left( \frac{P(0)}{1-P(0)} \right) = t \]

\[ P(t) \frac{P(t)}{1-P(t)} = e^t \]

\[ P(t) = e^t \times \frac{P(0)}{1-P(0)} \]

\[ P(t) = \frac{e^t \frac{P(0)}{1-P(0)}}{1 + e^t \frac{P(0)}{1-P(0)}} \]

\[ P(t) = \frac{P(0) e^t}{1-P(0) + P(0) e^t} \]
Individual infection is persistent:
- Assume all infected individuals remain infectious
- $S \rightarrow I$ model ($S$ for “Susceptible”, $I$ for “Infected”)

Growth of infected populations (denoted by $y$):
- Reproduction rate: infection probability $\beta$
- Resources: non-infected nodes (e.g. $n-y$)

\[
\frac{dy}{dt} = \beta y(t) (1 - y(t))
\]
Assume that infection is temporary and recurrent
- An infected node goes through an infectious period (equivalently, becomes non-infectious with rate $\gamma$)
- After the period, it is again susceptible

Evolution of $\frac{dy}{dt} = \beta y (A - y) - \gamma y$

- $\beta > \gamma$: endemics $\lim y = (1 - \gamma/\beta)$
- $\beta < \gamma$: epidemics dies $\lim y = 0$
Assume that infection is temporary and transient

- An infected node goes through an infectious period (equivalently, becomes non-infectious with rate $\gamma$)
- After the period, it is removed (vaccinated or dead)

Evolution follows similar equations

\[
\begin{align*}
\frac{dx}{dt} &= -\beta y x \\
\frac{dy}{dt} &= \beta y x - \delta y \\
\frac{dz}{dt} &= \delta y
\end{align*}
\]
Epidemic model #3: $S \rightarrow I \rightarrow R$

* Consequences of the dynamics
  - Infected individuals disappear: $\lim y = 0$
  - Number of removed individuals ($\lim_{t \to \infty} z(t)$) satisfies
    \[ \lim_{t \to \infty} z = x(0) \exp(-\beta \lim z / \gamma) \]
A general methodology

\{A, \ldots, L\}

* A large population of nodes in different states (S,I)
  - Dynamics only depends on fraction of nodes in state

* A condition on the rate of transitions
  - \( f^N(\text{measure})/\varepsilon(N) \) converges to \( f(\text{measure}) \)
  - Thm: \( M(t \times \varepsilon(N)) \) converges to solution \( \frac{dm}{dt} = f(m) \)
Continuous epidemics: summary

<table>
<thead>
<tr>
<th>Type</th>
<th>Outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>S→I</td>
<td>Everyone infected</td>
</tr>
<tr>
<td>S↔I</td>
<td>Depends on infection/recovery rate</td>
</tr>
<tr>
<td>S→I→R</td>
<td>No infectious node</td>
</tr>
</tbody>
</table>

Follows differential equation:
- Initial conditions: Fraction already infected
- Outcomes depend on type
- Cvg exponentially fast
- Mean field limit of discrete
  - No topology